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Heat transfer in higher-order boundary layer flows at low Prandtl number with suction and injection

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SUMMARY
In this paper the heat transfer in the second order boundary layer flow of an incompressible fluid with uniform oncoming stream is studied at low Prandtl numbers using the method of matched asymptotic expansions. The suction and injection are also included in first as well as second-order problems. For each of the second-order effects due to longitudinal curvature, transverse curvature, displacement and suction or injection the first five terms in an asymptotic expansion are obtained and compared with available exact results.

1. Introduction

It is well known that there are a large number of flow situations which cannot be dealt with in the framework of classical or Prandtl's boundary-layer theory. For such more complicated flows more accurate solutions of Navier-Stokes' equations than that of classical boundary-layer theory are needed. One such class of problems is governed by what is known as higher-order boundary layer theory [1].

The classical boundary layer theory of Prandtl forms the first term in an asymptotic expansion of the Navier-Stokes equation for large Reynolds number. The second term of the asymptotic expansion (of the order $R^{-\frac{1}{2}}$) includes the effects of curvature and interaction of the boundary layer with the external flow such as displacement and vorticity, provided the parameters representing these effects are of order unity. The second-order boundary-layer theory has received attention from many authors and a critical review has been recently given by Van Dyke [2].

The objective of the present work is to calculate the heat transfer at small values of the Prandtl number, due to second-order effects in two-dimensional and axisymmetric incompressible fluid flows with suction and injection. The free stream is assumed to be uniform and at zero angle of attack. The problem is of interest in heat transfer in liquid metals. The utility of such an analysis to gas flow is discussed in [5]. Further, it is well known that studies of such limiting solutions for small Prandtl number are useful even if the Prandtl number does not really has limiting values. The method of matched asymptotic expansions is employed to obtain low Prandtl number solutions that include the effects of longitudinal curvature, transverse curvature, displacement and first and second-order suction. For each of the second order effects, the first five terms in the asymptotic expansion for small Prandtl number has been evaluated. For the first order problem with zero suction, the small Prandtl number solutions have been studied by Morgan, Pipkin and Warner [3], Goddard and Acrivos [4] and Narasimha and Afzal [5].

2. Governing equations

The Navier-Stokes equations for a steady, plane and axisymmetric flow of an incompressible fluid in the usual nondimensional notation are [1]

$$\operatorname{div} \boldsymbol{U} = 0, \tag{1}$$

$$\boldsymbol{U}\cdot \operatorname{grad} U + \operatorname{grad} P = -R^{-1} \operatorname{curl} \operatorname{curl} \boldsymbol{U}, \tag{2}$$

$$\boldsymbol{U}\cdot \operatorname{grad} T - (\sigma R)^{-1}\nabla^2 T = 0. \tag{3}$$

Here $\boldsymbol{U} = (u, v)$ is the velocity vector, P the static pressure, T the static temperature, R the characteristic Reynolds number and σ the Prandtl number. The boundary conditions are: at the wall ($n = 0$)

$$T = T_w,\ u = 0,\ v = R^{-\frac{1}{2}}v_{w1}(s) + R^{-1}v_{w2}(s) + \ldots, \tag{4a}$$

and far away from the body

$$\boldsymbol{U} = \boldsymbol{i},\ T = T_\infty, \tag{4b}$$

where vector $\boldsymbol{i}$ is the unit vector in the direction of free stream.

The outer expansions for the variables are [1]

$$\left.\begin{aligned} u &= U_1(s, n) + R^{-\frac{1}{2}}U_2(s, n) + \ldots,\\ v &= V_1(s, n) + R^{-\frac{1}{2}}V_2(s, n) + \ldots,\\ p &= P_1(s, n) + R^{-\frac{1}{2}}P_2(s, n) + \ldots,\\ T &= T_1(s, n) + R^{-\frac{1}{2}}T_2(s, n) + \ldots. \end{aligned}\right\} \tag{5}$$

Substituting these expansions in the Navier-Stokes equations and collecting the coefficients of same order, the first term gives the well-known Euler equations and the second term the equations for displacement flow.

The inner expansions written in terms of the inner (Prandtl) variable

$$N = n\sqrt{R}$$

are as follows [1]

$$\left.\begin{aligned} u &= u_1(s, N) + R^{-\frac{1}{2}}u_2(s, N) + \ldots,\\ v &= R^{-\frac{1}{2}}v_1(s, N) + R^{-1}v_2(s, N) + \ldots,\\ p &= p_1(s, N) + R^{-\frac{1}{2}}p_2(s, N) + \ldots,\\ T &= t_1(s, N) + R^{-\frac{1}{2}}t_2(s, N) + \ldots. \end{aligned}\right\} \tag{6}$$

Here the first term gives the equations for Prandtls boundary layer and the second term gives the corrections needed at moderately large Reynolds number. Matching the inner and outer expansions in an overlap domain the additional boundary conditions needed for closure are obtained.

Introducing the following similarity variables

$$\xi = \int_0^s U_1(s, 0) r^{2j} ds, \quad \eta = N U_1(s, 0) r^j / \sqrt{(2\xi)} \tag{7a, b}$$

and the expressions for first and second order stream functions and temperature

$$\psi_1 = \sqrt{(2\xi)} f(\eta), \quad t_1(s, N) = T_\infty + (T_w - T_\infty) g(\eta), \tag{8a, b}$$

$$\psi_2 = \sqrt{(2\xi)} F(\eta), \quad t_2(s, N) = (T_w - T_\infty) G(\eta), \tag{9a, b}$$

the first and second-order boundary layer equations reduce to the following [6, 7].

First order boundary-layer equations:

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{10}$$

$$f(0) = C_1, \ f'(0) = 0, \ f'(\infty) = 1, \tag{11a, b, c}$$

$$g'' + \sigma f g' = 0, \tag{12}$$

$$g(0) = 1, \ g(\infty) = 0. \tag{12a, b}$$

Here β, the well-known Falkner-Skan pressure gradient parameter and $C_1 = -\sqrt{2\xi} v_{w1}/U_1$ the first order suction at the wall are assumed to be constants.

Second-order boundary layer equations:

$$\begin{aligned} F''' + fF'' - 2\beta f'F' + f''F = {} & k_l[-\eta(\beta + 1) f''' + (\beta - 1)(f'' + ff') \\ & + 2\beta(\beta\eta + \alpha)]/(\beta + 1) + k_t[-\eta(2\beta + f''') + f'' + ff'] - 2\beta D, \end{aligned} \tag{13}$$

$$F(0) = C_2, \ F'(0) = 0, \tag{14a, b}$$

$$F'(\eta) \sim -k_l \eta + k_t \eta + D \ \text{as} \ \eta \to \infty, \tag{14c}$$

$$G'' + \sigma f G' + \sigma F g' = -(k_l + k_t)(\eta g')', \tag{15}$$

$$G(0) = G(\infty) = 0. \tag{16a, b}$$

In the above equations the terms proportional to k_l arise because of the longitudinal curvature, k_t due to transverse curvature, D due to displacement and C_2 due to second order suction or injection. These parameters are defined as

$$\left.\begin{aligned} k_l &= \sqrt{(2\xi)} K / [r^j U_1(s, 0)], \\ k_t &= \sqrt{(2\xi)} j \cos\theta / [r^{j+1} U_1(s, 0)], \\ D &= U_2(s, 0)/U_1(s, 0), \\ C_2 &= -\sqrt{2\xi}\, v_{w2}/U_1(s, 0), \end{aligned}\right\} \tag{17}$$

and should be constant for jointly self-similar flows. Further, the quantity α is defined by

$$\alpha = \lim_{\eta \to \infty} (\eta - f). \tag{18}$$

The momentum and energy equations for the first and second-order problems are not coupled. Hence for studying their solutions at low σ the solution of the momentum equations may be assumed to be known [6, 7]. We need their solutions for large η:

$$f = \eta - \alpha + 0(\eta^{-\infty}), \tag{19a}$$

$$F = -k_l\eta^2/2 + k_t\eta^2/2 + D\eta - \delta + 0(\eta^{-\infty}). \tag{19b}$$

Here the symbol $0(\eta^{-\infty})$ denotes exponentically small terms in the limit as $\eta \to \infty$.

3. First-order solutions at low Prandtl number

The low σ analysis of the first-order boundary-layer equations (12) for $\beta = 0 = C_1$ has has been made by Narasimha and Afzal [4]. We here first extend their analysis for arbitrary values of β and C_1 and then proceed to the study of second-order equations.

Following Narasimha and Afzal we study two limits: an inner (defined as $\sigma \to 0$ with η fixed) and an outer ($\zeta = \sigma^{\frac{1}{2}}\eta$ fixed as $\sigma \to 0$), along with two corresponding asymptotic expansions and match them in an overlap domain.

The inner expansion is

$$g = \sum_{m=0} g_m \sigma^{m/2} \tag{20}$$

and the corresponding inner equations are

$$g_0'' = g_1'' = 0, \; g_n'' = -fg_{n-2}', \qquad n = 2, 3, 4, \ldots . \tag{21}$$

The solutions to the first five equations which satisfy the boundary conditions at the wall are

$$\left.\begin{aligned} g_0 &= a_0\eta + 1, \; g_1 = a_1\eta, \\ g_2 &= -a_0[\eta^3/6 - \alpha\eta^2/2 + \eta I_0(\eta) - I_1(\eta)] + a_2\eta, \\ g_3 &= -a_0[\eta^3/6 - \alpha\eta^2/2 + \eta I_0(\eta) - I_1(\eta)] + a_3\eta, \\ g_4 &= -a_0[\eta^5/40 - \alpha\eta^4/8 + \alpha^2\eta^3/6 + (\eta^3/6 - \alpha\eta^2/2)I_0(\eta) \\ &\quad - I_3(\eta)/6 + \alpha I_2(\eta)/2 + \eta I_{00}(\eta) - I_{10}(\eta)] \\ &\quad - a_2[\eta^3/6 - \alpha\eta^2/2 + \eta I_0(\eta) - I_1(\eta)] + a_4\eta. \end{aligned}\right\} \tag{22}$$

Here $I_m(\eta)$ and $I_{mn}(\eta)$ are certain integrals defined by

$$I_m = \int_0^\eta \eta_1^m f^*(\eta_1)\, d\eta_1, \tag{23}$$

$$I_{mn} = \int_0^\eta \eta_1^m f^*(\eta_1) I_n(\eta_1)\, d\eta_1, \tag{24}$$

and

$$f^*(\eta) = \alpha - \eta + f(\eta) \tag{25}$$

is exponentially small as $\eta \to \infty$. These integrals are bounded as $\eta \to \infty$.

The above solution (22) do not satisfy the boundary condition at infinity and is singular there. This singularity is similar to one encountered in improving Stokes' solution for low Reynolds number flow past a circular cylinder (Van Dyke [8]).

In the outer limit ($\zeta = \sigma^{\frac{1}{2}}\eta$ fixed as $\sigma \to 0$) valid for large η the energy equation (12) using (19a) yields:

$$g_{\zeta\zeta} + (\zeta - \sigma^{\frac{1}{2}}\alpha)g_{\zeta} = 0(\sigma^{\infty}), \tag{26}$$

an equation which is correct to all orders in σ, i.e. the error is exponentially small. This is due to the fact that the thermal boundary layer is much thicker than the momentum boundary layer and all that the momentum layer does far away is to displace the stream lines from their inviscid position by an amount α. The outer equations of all the orders in σ proceed from this simple equation which can be solved once and for all. If

$$Z = \zeta - \sigma^{\frac{1}{2}}\alpha \tag{27}$$

the equation (26) reduces to

$$g_{ZZ} + Zg_{Z} = 0. \tag{28}$$

The solution to this equation satisfying the boundary condition at infinity is

$$g = -b\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \operatorname{erf}(Z/\sqrt{2}) \tag{29}$$

where $\operatorname{erf}(x)$ is the well-known error function, defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{30}$$

Further, $b = b(\sigma)$ is a constant of integration independent of Z, but a function of σ, say:

$$b = \sum_{m=0} b_m \sigma^{m/2}. \tag{31}$$

Matching the inner and outer expansions (20) and (31) in the overlap domain we get

$$b = 1 - \sqrt{\frac{2\sigma}{\pi}} + \frac{2\alpha^2\sigma}{\pi} - \sqrt{\frac{2\sigma^3}{\pi}}\left[I_0(\infty) + \frac{12 - \pi}{6\pi}\alpha^3\right]$$
$$+ \frac{4\alpha\sigma^2}{\pi}\left[I_1(\infty) + \frac{12 - \pi}{12\pi}\alpha^3\right] + 0(\sigma^{\frac{5}{2}}) \tag{32}$$

and the heat transfer rate at the wall is

$$t_1'(0)/(T_\infty - T_w) = \sqrt{\frac{2\sigma}{\pi}} - \frac{2\alpha\sigma}{\pi} + \sqrt{\frac{2\sigma^3}{\pi}}\left[I_0(\infty) + \frac{4 - \pi}{4\pi}\alpha^2\right]$$
$$- \frac{2\sigma^2}{\pi}\left[\alpha I_0(\infty) + I_1(\infty) - \frac{2(\pi - 3)}{3\pi}\alpha^3\right] + 0(\sigma^{\frac{5}{2}}). \tag{33}$$

A uniformly valid solution, to the lowest order, can easily be written down by subtracting the common part from the union of the inner and outer solutions and will not be given here to save space.

4. Second-order solutions at low Prandtl number

Let us first consider the second-order momentum equation. Its solution, needed later, valid for large η, is

$$F = -k_l\eta^2/2 + k_t\eta^2/2 + D\eta - \delta + 0(\eta^{-\infty}). \tag{34}$$

It is advantageous to define a function

$$F^* = \delta - D\eta + k_l\eta^2/2 - k_t\eta^2/2 + F(\eta) \tag{35}$$

which is bounded throughout the domain: for large η it is exponentially small and for $\eta = 0$ its value is $\delta + C_2$.

We now study the second-order energy equation for $\sigma \to 0$. Here again two limits and the two corresponding asymptotic expansions are studied. An inspection of the first-order solution suggests the following inner expansion

$$G = \sum_{m=0} G_m \sigma^{m/2}. \tag{36}$$

Substituting the inner expansions (20) and (36) in the energy equation (15) and collecting the coefficients of like powers of $\sigma^{\frac{1}{2}}$, we get

$$\begin{aligned} G_m'' &= -(k_l + k_t)g_m', \quad m = 0, 1 \\ G_m'' &= -(k_l + k_t)g_m' + [(k_l + k_t)\eta f - F]g_{m-2}' - fG_{m-2}, \quad m = 2, 3, 4, \ldots . \end{aligned} \tag{37}$$

Integration of the first five of the above equations which satisfy the boundary conditions at the wall yields

$$\begin{aligned} G_0 &= A_0\eta,\ G_1 = -(k_l + k_t)a_1\eta^2/2 + A_1, \\ G_2 &= -A_0\eta^3/6 + [A_0\alpha - (k_l + k_t)a_1]\eta^2/2 + [A_2 - A_0(\eta)]\eta + A_0 I_1(\eta), \\ G_3 &= a_1(6k_l - 4k_t)\eta^4/24 - [A_1 + a_1 D + 3a_1(k_l + k_t)]\eta^3/6 \\ &\quad + [A_1\alpha + a_1\delta + (k_l + k_t)\{a_1 I_0(\eta) - a_3\}]\eta^2/2 \\ &\quad + [A_3 - A_1 I_0(\eta) - a_1 J_0(\eta) + (k_l + k_t)a_1 I_1(\eta)]\eta \\ &\quad + A_1 I_1(\eta) + a_1 J_1(\eta) - 3(k_l + k_t)a_1 I_2(\eta)/2, \\ G_4 &= A_0\eta^5/40 + [-3A_0\alpha + a_2(6k_l + 4k_t)]\eta^4/24 \\ &\quad + [-A_2 + A_0\alpha^2 - 3a_2\alpha(k_l + k_t) - a_2 D + A_0 I_0(\eta)]\eta^3/6 \\ &\quad + [A_2\alpha + a_2\delta - a_4(k_l + k_t) + \{a_2(k_l + k_t) - A_0\alpha\}I_1(\eta)]\eta^2/2 \\ &\quad + [A_4 + a_2\{(k_l + k_t)I_1(\eta) - J_0(\eta)\} + A_0 I_{00}(\eta) + A_2 I_0(\eta)]\eta \\ &\quad + A_2 I_1(\eta) - A_0 I_3(\eta)/6 - A_0 I_{10}(\eta) + a_2 J_1(\eta) \\ &\quad + [A_0\alpha - 3(k_l + k_t)a_2]I_2(\eta)/2. \end{aligned} \tag{38}$$

Here A_m's are constant of integration to be determined and $J_m(\eta)$ is a bounded integral defined by

$$J_m(\eta) = \int_0^\eta \eta_1^m F^*(\eta_1)\, d\eta_1. \tag{39}$$

Its behaviour for large η is

$$J_m(\eta) = J_m(\infty) + 0(\eta^{-\infty}) \text{ as } \eta \to \infty. \tag{40}$$

The inner solution (38) not only fails to satisfy the boundary condition at infinity but is singular there. To analyse the nature of the singularity at large η, an order of magnitude analysis of the equation (15) employing the outer variable $\zeta = \sigma^{\frac{1}{2}}\eta$ is carried out. It leads to the following outer variable

$$\chi(\zeta) = \sigma^{\frac{1}{2}}G(\eta)$$

and the outer limit as χ, ζ fixed as $\sigma \to 0$. Using solution (35) of the second-order momentum equation for large η, the second-order energy equation in the outer limit yields

$$\begin{aligned}\chi_{\zeta\zeta} &+ (\zeta - \sigma^{\frac{1}{2}}\alpha)\chi_\zeta \\ &= g_\zeta[(3k_l + k_t)\zeta^2/2 - \sigma^{\frac{1}{2}}\zeta(D + \alpha k_l + \alpha k_t) - k_l - k_t + \delta\sigma] + 0(\sigma^\infty).\end{aligned} \tag{41}$$

The above equation is also correct to all orders in σ, i.e. the error is exponentially small as $\sigma \to 0$. Introducing the variable Z we get

$$\begin{aligned}\chi_{ZZ} + Z\chi_Z = g_Z[&(3k_l + k_t)Z^2/2 + \sigma^{\frac{1}{2}}Z(2k_l\alpha - D) \\ &+ k_l(\sigma\alpha^2/2 - 1) - k_t(1 + \sigma\alpha^2/2) + \sigma(\delta - D\alpha)].\end{aligned} \tag{42}$$

The solution of the above equation, which satisfies the boundary condition at infinity is

$$\begin{aligned}\chi = &-\frac{B}{\sqrt{2}}\Gamma(\tfrac{1}{2}, Z^2/2) + b[k_l(1 - \sigma\alpha^2/2) + k_t(1 + \sigma\alpha^2/2) + \sigma(D\alpha - \delta)] \\ &\times \Gamma(1, Z^2/2) + \frac{\sigma^{\frac{1}{2}}b}{\sqrt{2}}(D - 2k_l\alpha)\Gamma(\tfrac{3}{2}, Z^2/2) - b(k_l + k_t/3)\Gamma(2, Z^2/2),\end{aligned} \tag{43}$$

where B is a constant of integration independent of Z but a function of σ, say:

$$B = \sum_{m=0} B_m\sigma^{m/2}. \tag{44}$$

The function $\Gamma(m, x)$ is the incomplete gamma function defined by

$$\Gamma(m, x) = \int_x^\infty e^{-z}z^{m-1}\,dz. \tag{45}$$

Its asymptotic behaviour for small values of x, needed for matching is

$$\Gamma(m, x) = \Gamma(m) - \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(m + n)\Gamma(n + 1)}\right]x^{m+n}. \tag{46}$$

Matching the inner and outer solutions we get

$$\begin{aligned}B = &-\frac{4k_t}{3\pi} + \sqrt{\frac{2\sigma}{\pi}}\left[\frac{8}{3\pi}k_t\alpha + k_l\alpha - \frac{D}{2}\right] - \frac{2\sigma}{\pi}\left[\frac{4}{\pi}k_t\alpha^2 + k_l\alpha^2 - \delta\right] \\ &- \sqrt{\frac{2\sigma^3}{\pi}}\left[\frac{8k_t}{3\pi}\left\{I_1(\infty) + \frac{24 - \pi}{6\pi}\alpha^3\right\} + \frac{2}{\pi}k_l\alpha^3 + \frac{D\alpha^2}{\pi} - \frac{4\delta\alpha}{\pi}\right] \\ &+ \frac{2\sigma^2}{\pi}\left[J_1(\infty) - k_l\left\{\tfrac{3}{2}I_2(\infty) + \alpha I_1(\infty) + \frac{12 - \pi}{6\pi}\alpha^4\right\} + \right.\end{aligned}$$

$$+ k_t\left\{-\tfrac{3}{2}I_2(\infty) + \frac{\pi - 8}{\pi} I_1(\infty) + \frac{\alpha^4}{36}(3\pi^2 + 40\pi - 480)\right\}$$

$$+ D\left\{I_1(\infty) - \frac{12-\pi}{6\pi}\alpha^3\right\} + \delta\alpha^2\frac{12-\pi}{2\pi}\Bigg] + 0(\sigma^{\frac{5}{2}}). \quad (47)$$

The temperature gradient at the wall is

$$G'(0) = \sum_{m=0} A_m\sigma^{m/2}$$

$$= -\frac{4}{3\pi}k_t + \sqrt{\frac{2\sigma}{\pi}}\left[k_t\alpha\frac{8-3\pi}{3\pi} - \frac{D}{2}\right]$$

$$+ \sigma\left[\frac{2\delta}{\pi} - \frac{4}{3\pi}k_t\left\{I_0(\infty) + 2\alpha^2\frac{3-\pi}{\pi}\right\}\right] + \sqrt{\frac{2\sigma^3}{\pi}}\Bigg[-J_0(\infty)$$

$$+ \frac{k_t}{3\pi}\left\{(8-3\pi)\alpha I_0(\infty) + (8+3\pi)I_1(\infty) + \frac{\alpha^3}{6\pi}(3\pi^2 - 68\pi + 192)\right\}$$

$$+ k_l I_1(\infty) + D\left\{-\tfrac{1}{2}I_0(\infty) + \frac{4-\pi}{4\pi}\alpha^2\right\} - \frac{4-\pi}{\pi}\delta\alpha\Bigg]$$

$$+ \frac{2}{\pi}\sigma^2\Bigg[J_1(\infty) + \alpha J_0(\infty)$$

$$+ \frac{2k_t}{3}\left\{-\tfrac{9}{4}I_2(\infty) + \frac{(\pi-8)}{\pi}\alpha I_1(\infty) + I_{00}(\infty) - I_0^2(\infty)\right.$$

$$\left.+ \frac{6-\pi}{\pi}\alpha^2 I_0(\infty) - \frac{\alpha^4}{6\pi}(3\pi^2 - 46\pi + 120)\right\}$$

$$- k_l\{\tfrac{3}{2}I_2(\infty) + \alpha I_1(\infty)\} + D\left\{I_1(\infty) + \frac{2(\pi-3)}{3\pi}\alpha^3\right\}$$

$$+ \delta\left\{I_0(\infty) + \frac{6-2\pi}{\pi}\alpha^2\right\}\Bigg] + 0(\sigma)^{\frac{5}{2}}. \quad (48)$$

5. Decomposition of second-order heat transfer results

It may be noted that the first-order boundary-layer equations are nonlinear while the second-order boundary-layer equations are linear. Thus the second-order effects can be linearly separated out into several effects, each associated with a particular interpretation. In the present study these can be divided into terms representing the transverse curvature, longitudinal curvature, displacement and second-order suction. Thus we set

$$F = k_l F_l(\eta) + k_t F_t(\eta) + DF_d(\eta) + C_2 F_c(\eta), \quad (49)$$

$$t_2 = k_l t_{2l}(\eta) + k_t t_{2t}(\eta) + Dt_{2d}(\eta) + C_2 t_{2c}(\eta), \quad (50)$$

$$\delta = k_l\delta_l + k_t\delta_t + D\delta_d + C_2\delta_c. \quad (51)$$

The equations for F_l etc. can easily be written down and will not be given here. Decomposing the expression for second-order heat transfer (48) into the above four components we get the following:

Transverse curvature:

$$t'_{2t}(0)/(T_\infty - T_w) = \frac{4}{3\pi} - \sqrt{\frac{2\sigma}{\pi}}\,\frac{8-3\pi}{3\pi}\alpha + \sigma$$

$$\times\left[-\frac{2\delta_t}{\pi} + \frac{4}{3\pi}\left\{I_0(\infty) + \frac{6-2\pi}{\pi}\alpha^2\right\}\right] - \sqrt{\frac{2\sigma^3}{9\pi^3}}\left[-3\pi J_{0t}(\infty) + (8-3\pi)\alpha I_0(\infty)\right.$$

$$\left. + (8+3\pi)I_1(\infty) + \frac{\alpha^3}{6\pi}(3\pi^2 - 68\pi + 192) - 3(4-\pi)\alpha\delta_t\right]$$

$$- \frac{4\sigma^2}{3\pi}\left[\tfrac{3}{2}\{\alpha J_{0t}(\infty) + J_{1t}(\infty)\} - \tfrac{9}{4}I_2(\infty) + \frac{\pi-8}{\pi}\alpha I_1(\infty)\right.$$

$$+ I_{00}(\infty) - I_0^2(\infty) + \frac{6-2\pi}{\pi}\alpha^2 I_0(\infty) - \frac{\alpha^4}{24}(12\pi^2 - 18\pi + 480)$$

$$\left. + 3\delta_t\left\{I_0(\infty) + \frac{3-\pi}{3\pi}\alpha^2\right\}\right] + 0(\sigma^{\frac{5}{2}}). \tag{52}$$

Here

$$J_{mt}(\eta) = \int_0^\eta \eta^m(\delta_t - \eta^2/2 + F_t)\,d\eta, \tag{53a}$$

$$\delta_t = \lim_{\eta\to\infty}(\eta^2/2 - F_t). \tag{53b}$$

Longitudinal curvature:

$$t'_{2l}(0)/(T_\infty - T_w) = -\frac{2\delta_l\sigma}{\pi} + \sqrt{\frac{2\sigma^3}{\pi}}\left[J_{0l}(\infty) - I_1(\infty) + \frac{4-\pi}{\pi}\alpha\delta_l\right]$$

$$+ \frac{2\sigma^2}{\pi}\left[J_{1l}(\infty) + \alpha J_{0l}(\infty) - \tfrac{3}{2}I_2(\infty) - \alpha I_1(\infty)\right.$$

$$\left. + \delta_l\left\{I_0(\infty) + \frac{6-2\pi}{\pi}\alpha^2\right\}\right] + 0(\sigma^{\frac{5}{2}}). \tag{54}$$

Here

$$J_{ml}(\eta) = \int_0^\eta (\delta_l + \eta^2/2 + F_l)\,d\eta, \tag{55a}$$

$$\delta_l = \lim_{\eta\to\infty}(-\eta^2/2 - F_l). \tag{55b}$$

Displacement:

$$t'_{2d}(0)/(T_\infty - T_w) = \sqrt{\frac{2\sigma}{\pi}} - \frac{2\delta_d}{\pi}\sigma + \sqrt{\frac{2\sigma^3}{\pi}}\Big[J_{0d}(\infty) + \tfrac{1}{2}I_0(\infty)$$

$$+ (4\alpha\delta_d - \alpha^2)\frac{4-\pi}{4\pi}\Big] - \frac{2\sigma^2}{\pi}\Big[J_{1d}(\infty) + \alpha J_{0d}(\infty) + I_1(\infty)$$

$$+ \delta_d I_0(\infty) + 2(3\delta_d\alpha^2 - \alpha^3)\frac{3-\pi}{3\pi}\Big] + 0(\sigma^{\frac{5}{2}}). \tag{56}$$

Here

$$J_{md}(\eta) = \int_0^\eta \eta^m(\delta_d - \eta + F_d)\,d\eta, \tag{57a}$$

$$\delta_d = \lim_{\eta\to\infty}(\eta - F_d). \tag{57b}$$

It may be noted that for the special case when first-order suction is absent ($C_1 = 0$) the momentum and energy equations admit the following closed form solution

$$F_d = \tfrac{1}{2}(\eta f' + f), \quad G_d = \tfrac{1}{2}\eta g'. \tag{58a, b}$$

The heat transfer rate is given by

$$G'_d(0) = \tfrac{1}{2}g'(0). \tag{58c}$$

It can be easily shown that for this special case ($C_1 = 0$)

$$\delta_d = \alpha/2,\ J_{0d}(\infty) = 0,\ J_{1d}(\infty) = -I_1(\infty)/2, \tag{59}$$

and the result (56) reduces to that given by the closed form expression (58c).

Second-order suction and injection:

$$t'_{2c}(0)/(T_\infty - T_w) = -\frac{2\delta_c}{\pi}\sigma + \sqrt{\frac{2\sigma^3}{\pi}}\Big[J_{0c}(\infty) + \frac{4-\pi}{\pi}\delta_c\alpha\Big]$$

$$-\frac{2\sigma^2}{\pi}\Big[J_{1c}(\infty) + \alpha J_{0c}(\infty) + \delta_c I_0(\infty) + \frac{2(3-\pi)}{\pi}\delta_c\alpha^2\Big] + 0(\sigma^{\frac{5}{2}}). \tag{60}$$

Here

$$J_{mc}(\eta) = \int_0^\eta \eta^m(\delta_c + F_c)\,d\eta, \tag{61a}$$

$$\delta_c = -F_c(\infty). \tag{61b}$$

6. Discussion

In the earlier sections the contributions to heat transfer, to the order σ^2, due to transverse curvature, longitudinal curvature, displacement and suction/injection are obtained. The leading terms, at low σ, for the first order (classical) boundary layer behaves like $\sigma^{\frac{1}{2}}$ while

TABLE 1

Values of integrals evaluated from solutions of first and second-order momentum equations

β	1.0	0.5	0.0	−0.1
α	0.64790	0.80455	1.21678	1.44270
$I_0(\infty)$	0.35951	0.53153	1.09143	1.45539
$I_1(\infty)$	0.17711	0.30305	0.79617	1.16560
$I_2(\infty)$	0.15831	0.30733	0.99467	1.57624
$I_{00}(\infty)$	0.06461	0.14126	0.59581	1.05907
δ_l	0.73684	1.06503	2.10870	2.83740
$J_{0l}(\infty)$	0.91760	1.56004	4.12306	0.19394
$J_{1l}(\infty)$	0.74227	1.44365	4.79839	7.85781
δ_t	0.17400	0.25981	0.49352	0.54548
$J_{0t}(\infty)$	0.18880	0.32954	0.76156	0.94699
$J_{0t}(\infty)$	0.14172	0.28260	0.83746	1.06955
δ_d	0.32395	0.40227	0.60839	0.72135
$J_{0d}(\infty)$	0.0	0.0	0.0	0.0
$J_{1d}(\infty)$	−0.08858	−0.15153	−0.39806	−0.58278
δ_c	−1.23713	−1.38287	−2.05434	−2.73510
$J_{0c}(\infty)$	−0.24581	−0.46142	−1.66344	−3.03345
$J_{1c}(\infty)$	−0.17477	−0.36643	−1.65322	−3.28597

TABLE 2

Coefficients for heat transfer series $G_i'(0) = \sum_{m=0} A_{im}\sigma^{m/2}$, $i = t, l, d, c$ when $C_1 = 0$ *for various second-order effects*

i	β	A_0	A_1	A_2	A_3	A_4
Transverse curvature	1.0	0.4244	0.0782	0.0257	−0.08197	0.07619
	0.5	0.4244	0.0970	0.0343	−0.1217	0.1477
	0.0	0.4244	0.1468	0.0924	−0.2498	0.7009
	0.1	0.4244	0.1740	0.1909	−0.4611	1.680
Longitudinal curvature	1.0	0.0	0.0	−0.469	0.6345	−0.777
	0.5			−0.6780	1.238	−1.5902
	0.0			−1.3424	3.4827	−5.9680
	−0.1			−1.8063	5.8696	−10.406
Displacement	1.0	0.0	0.3989	−0.2062	0.2904	−0.2058
	0.5		0.3989	−0.2561	0.2224	−0.2276
	0.0		0.3989	−0.3873	0.5937	−0.8159
	−0.1		0.3989	−0.4592	0.9394	1.0107
Suction and injection	1.0	0.0	0.0	0.7878	−0.2695	0.4661
	0.5			0.8252	−0.5252	0.8862
	0.0			1.3078	−2.1341	3.5939
	−0.1			1.7412	−4.2109	7.0854

for the second-order effect of transverse curvature it is like σ^0, longitudinal curvature like σ, displacement like $\sigma^{\frac{1}{2}}$ and suction/injection like σ.

The various integrals in these heat transfer expressions are evaluated by generating the solutions of the first and second-order momentum equations when $C_1 = 0$ and $\beta = 1, 0.5, 0.0$ and -0.1 and their values are given in Table 1. To demonstrate the utility of the present low σ results, the heat transfer series for each of the second-order effects are evaluated and their coefficients are displayed in Table 2.

When σ is of order unity the series for $\beta = 1$ of the transverse curvature and displacement are slowly convergent while those of the longitudinal curvature and suction/injection are divergent. For $\sigma = 1$, the value of heat transfer $G'(0)$ due to transverse curvature is 0.522 and for displacement is 0.277. The corresponding exact results [6] are 0.501 and 0.285. For $\sigma = 1$ the longitudinal curvature series

$$-0.4691 + 0.6341 - 0.7777 + \ldots$$

and suction/injection series

$$0.7878 - 0.2695 + 0.4661 + \ldots$$

are divergent. Some information from these series can be extracted by improving the convergence through the Euler transformation (see Meksyn [9]) to yield for longitudinal curvature

$$-0.2345 + 0.0414 + 0.0056 + \ldots = 0.187$$

and for suction/injection

$$0.3639 + 0.1296 + 0.0893 + \ldots = 0.58$$

The exact results [6] for longitudinal curvature heat transfer for $\sigma = 1$ is 0.161. For the suction/injection case the exact result [7] is known for $\sigma = 0.7$ to be 0.455. The present series for $\sigma = 0.7$ is

$$0.5514 - 0.1578 + 0.2284 + \ldots$$

and the Eulerized version

$$0.2757 + 0.0984 + 0.0580 + \ldots$$

yields 0.431 comparable to exact value of 0.455. Further as β decreases the radius of convergence of these series (see Table 2) also decreases when compared to the case for $\beta = 1$. For these cases the results can also be improved by using the Euler transformation.

Thus from the above discussion it is clear that the low σ results are useful even when σ is of order unity.

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